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A. B. Coble, E. R. Hedrick, W. R. Longley (*Chairman*).

AN UNCOUNTABLE, CLOSED, AND NON-DENSE POINT SET EACH OF WHOSE COMPLEMENTARY INTERVALS ABUTS ON ANOTHER ONE AT EACH OF ITS ENDS*

BY R. L. MOORE

On page 92 of the 1907 edition of Hobson's *The Theory of Functions of a Real Variable*, and again on page 113 of the second edition of the same treatise, there occurs the following statement: †

"*A non-dense closed set is enumerable if its complementary intervals are such that every one of them abuts on another one at each of its ends.*"

To prove this statement, Hobson lets G denote the non-dense closed set in question and argues, in part, as follows:

"In this case, all the points of G are either end-points of adjacent intervals, or limiting points, on both sides, of a sequence of such end-points; unless a‡ or b‡ be a limiting point, in which case it belongs to G. The end-points have the same cardinal number as the rational numbers, since the set of intervals is enumerable. Moreover the external § points form a finite set, or an enumerable set; because to each such external point there corresponds an enumerable set of end-points of which it is the limiting point and in this correspondence *any one end-point can correspond to at most two limiting points, one on each side of it.*"

Just what is the meaning of the above italicized statement? If P is an external point, then, by definition, there exists a

* Presented to the Society, December 29, 1922.

† This statement will be called Proposition A.

‡ Here a and b apparently denote the end-points of some interval which contains the set G.

§ External points are defined by Hobson as points such as are not end-points of any contiguous interval but are limit points on both sides of such end-points.

sequence of end-points (of intervals complementary to G) such that P is the sequential limiting point of this sequence. But clearly there exist infinitely many such sequences and if X is an end-point of any interval whatsoever complementary to G then there is one sequence of such end-points which contains X and has P as its limiting point. In the correspondence imagined by Hobson does P correspond, then, to every such point X? If not, then apparently it is intended that for each point P there should be selected, in some manner or other, some particular sequence of end-points converging to P and that the points of this particular sequence only should be considered as corresponding to P. Even if this be done, however, what justification (if any) is there for the statement that, in the correspondence so established, any one end-point can correspond to at most two limit points, one on each side of it? Certainly it is possible that for three different external points P_1, P_2, P_3 there should be selected three sequences $P_{11}, P_{12}, P_{13}, \cdots$; $P_{21}, P_{22}, P_{23}, \cdots$; $P_{31}, P_{32}, P_{33}, \cdots$ such that (1) these sequences converge respectively to P_1, P_2, P_3, and their points correspond respectively to the points P_1, P_2, P_3, and (2) $P_{11} = P_{21} = P_{31}$. In this case the end-point P_{11} would correspond to three different limiting points, contrary to Hobson's statement. In fact there exists a non-dense closed point set G satisfying the hypothesis of Hobson's proposition and such that there is no way whatsoever of establishing the correspondence indicated by Hobson without having some end-point correspond to uncountably many limiting points. To see this, let K denote a non-dense perfect set of points lying on an interval AB and containing the points A and B. Consider the intervals A_1B_1, A_2B_2, A_3B_3, $\cdots$ which are complementary to K (with respect to AB). For every two positive integers n and m let P_{nm} and $\overline{P}_{nm}$ denote points between A_n and B_n situated so that $P_{nm}A_n = \overline{P}_{nm}B_n = A_nB_n/2m$. Let G denote the point set obtained by adding, to the point set K, all the P_{nm}'s and all the $\overline{P}_{nm}$'s.

The existence of this example disproves Proposition A.

THE UNIVERSITY OF TEXAS

TOTAL GEODESIC CURVATURE AND GEODESIC TORSION*

BY J. K. WHITTEMORE

In a paper on the gyroscope, presented to this Society April 23, 1921, Professor W. F. Osgood introduced the notion of the "bending" of a spherical curve. The "bending" is defined as the rate of turning, per unit length of arc of the curve, of the plane determined by the tangent to the curve and the normal to the surface. It is the purpose of this note to show that the bending of a curve on any surface is equal to

$$\sqrt{\frac{1}{\rho_g^2}+\frac{1}{\tau_g^2}},$$

where ρ_g and τ_g are the radii of geodesic curvature and geodesic torsion respectively. An expression, believed to be new, for the geodesic torsion of any curve of the surface is also derived. Since the rate of turning of the principal normal of a curve,

$$\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}},$$

where ρ and τ are the radii of curvature and torsion respectively, is called the total curvature of a curve,† it seems appropriate to replace the term "bending" by "total geodesic curvature."

Let Γ be any curve of a surface S and P be any point of Γ; let $\tilde{\omega}$ be the angle measured from the positive direction of the principal normal to Γ at P to the positive direction of the normal to S at P, the angle being measured toward the positive binormal; let the direction cosines of the positive direction of the normal to S be X, Y, Z, and those of the positive directions of principal normal and binormal be respectively l, m, n and λ, μ, ν. Then the direction cosines of the normal to the plane of the tangent to Γ and the normal to S are

* Presented to the Society, October 29, 1921.

† ENCYCLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, III, 3, 1, p. 82.

$$A = l \sin \bar{\omega} - \lambda \cos \bar{\omega},$$
$$B = m \sin \bar{\omega} - \mu \cos \bar{\omega},$$
$$C = n \sin \bar{\omega} - \nu \cos \bar{\omega},$$

and the total geodesic curvature, g, is given by

$$g^2 = \Sigma \left(\frac{dA}{ds}\right)^2 = \frac{\sin^2 \bar{\omega}}{\rho^2} + \left(\frac{1}{\tau} - \frac{d\bar{\omega}}{ds}\right)^2 = \frac{1}{\rho_g^2} + \frac{1}{\tau_g^2},$$

where s is the arc of Γ. The third member of the preceding equation is obtained by differentiating the values given for A, B, C and using the Frenet-Serret formulas.*

It is of interest to note the value of the total geodesic curvature for certain special curves of a surface. 1. The total geodesic curvature is numerically equal to the geodesic curvature when the geodesic torsion vanishes, that is, when Γ is tangent at P to a line of curvature of S; this occurs at all points of Γ when and only when Γ is a line of curvature; it is true in particular for all spherical curves, the case considered by Professor Osgood. 2. The total geodesic curvature is numerically equal to the geodesic torsion when the geodesic curvature vanishes, that is, when Γ osculates a geodesic line at P; this occurs at all points of Γ when and only when Γ is a geodesic line; for such a line total geodesic curvature, geodesic torsion and torsion are numerically equal. The total geodesic curvature vanishes for all points of a geodesic line of curvature, which is necessarily a plane curve. 3. If Γ is an asymptotic line, the total geodesic curvature is at every point equal to the total curvature. This fact is evident from the definitions of the two curvatures, and appears also from the equations, true for any asymptotic line,

$$\rho = \pm \rho_g, \qquad \tau = \tau_g.$$

There exist, however, on every surface curves other than the asymptotic lines such that at every point the total geodesic curvature is equal to the total curvature. Along such a curve the following equation must be satisfied:

$$\left(\frac{d\bar{\omega}}{ds}\right)^2 - \frac{2}{\tau}\frac{d\bar{\omega}}{ds} = \frac{\cos^2 \bar{\omega}}{\rho^2}.$$

* See Eisenhart, *Differential Geometry*, pp. 17, 132, 138.

The asymptotic lines are given by the solution, $\cos \tilde{\omega} = 0$. For a given surface with given parameters, u, v, this equation becomes a differential equation of the third order for v as a function of u. If on the other hand any curve is given, a "surface band," that is, the curve and the normal to a surface at each point of the curve, is determined by a solution $\tilde{\omega}$ of this equation such that the given curve has the required property on any surface containing the band. If, in particular, the given curve is plane, the equation gives for the surface band $\cos \tilde{\omega} = \operatorname{sech} \varphi$ where φ is the angle made by the tangent to the given plane curve with any fixed direction in the plane.

We now prove that the geodesic torsion of any curve Γ of S is given by

$$\frac{1}{\tau_g} = \sin \theta \frac{d\sigma}{ds},$$

where σ is the arc of the curve corresponding to Γ in the spherical representation of S and θ is the angle measured from the positive direction of Γ to the positive direction of the corresponding curve in the spherical representation, the angle being measured in the direction of rotation from the positive direction of the parametric curve (v), or v constant, on S to the positive direction of (u).

The direction cosines of the positive directions on Γ and on its spherical representation are respectively dx/ds, dy/ds, dz/ds and $dX/d\sigma$, $dY/d\sigma$, $dZ/d\sigma$. We find

$$\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{dX}{d\sigma} & \frac{dY}{d\sigma} & \frac{dZ}{d\sigma} \\ X & Y & Z \end{vmatrix} = \sin \theta.$$

Suppose the parametric system to be such that Γ is a curve (v) and that the system is orthogonal so that $F = 0$. We may then evaluate the determinant above, writing

$$\frac{dx}{ds} = \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u}, \qquad \frac{dX}{d\sigma} = \frac{ds}{d\sigma} \frac{1}{\sqrt{E}} \frac{\partial X}{\partial u},$$

using the formula given by Eisenhart * for $\partial X/\partial u$, replacing $-D'/H$ by $1/\tau_g$, and noting that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ X & Y & Z \end{vmatrix} = H.$$

We find

$$\frac{1}{\tau_g}\frac{ds}{d\sigma} = \sin\theta,$$

so proving the formula given. A somewhat simpler proof can be given for the numerical correctness of this result, that is, if the question of sign is disregarded. Certain properties of geodesic torsion appear clearly from the preceding formula. 1. If Γ is tangent to a line of curvature of S at P, the tangents to Γ at P and to the spherical representation of Γ at the corresponding point are parallel, and $\theta = 1/\tau_g = 0$. 2. If Γ is tangent to an asymptotic line of S at P, we have $\theta = \pm\pi/2$ and $1/\tau_g = \pm d\sigma/ds$. Since in the direction of an asymptotic line $d\sigma^2/ds^2 = -K$, where K is the total curvature of S, we have Enneper's theorem:† in the direction of an asymptotic line $1/\tau_g = \pm\sqrt{-K}$. 3. Since the spherical representation of a minimal surface is conformal, ‡ $1/\tau_g = \pm\sqrt{-K}\sin 2\psi$ for any curve on such a surface, where ψ is the angle of the tangent to the curve and the tangent to either line of curvature. If a minimal surface is applicable to a surface of revolution, every geodesic line of the minimal surface, which in the application corresponds to a meridian of the surface of revolution, cuts all the lines of curvature of one family at the same angle.§ For such a geodesic the torsion varies as $\sqrt{-K}$ since ψ is constant.

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* Loc. cit., pp. 116, 138.

† Eisenhart, loc. cit., pp. 140, 141.

‡ Loc. cit., p. 251.

§ This result was first given by E. Bour, *Théorie de la déformation des surfaces*, JOURNAL DE L'ÉCOLE POLYTECHNIQUE, vol. 39 (1862).

THE NAME "DIVERGENT" SERIES

BY F. CAJORI

James Gregory is rightly credited with the introduction of the name "convergent" series. According to Reiff,* whom I have followed on this matter in my writings, Gregory also introduced the name "divergent" series. When a few years ago a correspondent raised the question whether Gregory really did use the word "divergent," I was not able to answer definitely because I could not secure access to a copy of Gregory's *Vera Circuli et Hyperbolae Quadratura*, in which the term occurs, according to Reiff. At last, I have found and examined a copy (Patavia, 1668), in the Naval Observatory at Washington, D. C.; the phrase "divergent series" does *not* occur in it, nor in Gregory's *Geometriae Pars Vniversalis* (Patavia, 1668), although in the *Vera Circuli, etc.*, "convergent" is used in the form of an adjective, verb, or noun over a hundred times. This term did not meet with immediate acceptance, for in 1705 † and again in 1713 ‡ Leibniz used the words "advergens," "advergentia" to signify convergent and convergence. These words did not cling to the mathematical phraseology. As regards the term "divergent," it was Nicolaus Bernoulli § who in 1713 first used "divergens" and "divergentia seriei," as is correctly stated in Cantor || and in the ENCYCLOPÉDIE.¶

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* R. Reiff, *Geschichte der Unendlichen Reihen*, Tübingen, 1889, p. 16.

† Leibniz's letter to J. Hermann, of April 7, 1705. See *Leibnizens Mathematische Schriften* (Ed. C. I. Gerhardt), vol. IV, p. 272. See also BIBLIOTHECA MATHEMATICA, (3), vol. 5 (1904), p. 308.

‡ Leibniz's letter to Nicolaus Bernoulli, of June 28, 1713. See *Leibnizens Mathematische Schriften*, vol. III, p. 985.

§ Nicolaus Bernoulli's letter to Leibniz, April 7, 1713. See *Leibnizens Mathematische Schriften*, vol. III, p. 983.

|| M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. III (2d ed.), p. 369.

¶ ENCYCLOPÉDIE, vol. 1, 1907, p. 184, note 199.

A QUALITATIVE DEFINITION OF THE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS *

BY P. FRANKLIN

1. *Introduction.* The object of the present note, which was suggested by a remark of Professor Birkhoff as to possible definitions of the trigonometric functions, is to define the trigonometric and hyperbolic functions by properties which shall be simple in the sense of not involving any of the ideas of the calculus, and qualitative in the sense of not involving relations as definitely explicit as functional or differential equations.†

2. *Postulates.* The characteristic properties (or postulates) which we use in our definition are stated in terms of a linear family of functions depending on two parameters, and they so restrict the family that each of its members is of the form $rF(mx + t)$ or the sum of two such expressions, where $F(x)$ is the function we wish to define. In proving this we shall incidentally give explicit rules for constructing the function $F(x)$ in terms of the family. We assume as the characteristic properties:

I. The two-fold linear family of functions $AG(x) + BH(x)$ (where $G(x)$ and $H(x)$ are any two independent members of the family) is independent of the choice of origin and direction of the x-axis; i.e., it is identical, as a family, with that given by $AG(x + c) + BH(x + c)$ or $AG(-x) + BH(-x)$.

II. Some pair of independent members of the family, as $G(x)$ and $H(x)$, are functions continuous for at least one value of x.

To these we shall add one of the following:

III (a). No member of the family never vanishes.

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† For definitions of the trigonometric functions involving such relations, see Osgood, *Lehrbuch der Funktionentheorie*, 1920, vol. 1, pp. 571–591.

III (*b*). There exists a member of the family which never vanishes, and all members of the family which never vanish are linearly dependent on this one, i.e., constant multiples of it.

III (*c*). There exist two linearly independent members of the family which never vanish.

We shall show that for I, II and III (*a*) the family is $A \sin mx + B \cos mx$, and hence each member is of the form $r \sin (mx + t)$. For I, II and III (*b*) the family is $A + Bx$, and hence each member of the family is of the form $mx + t$. Finally, for I, II and III (*c*) the family is $A \sinh mx + B \cosh mx$, and hence each member is of the form $r \sinh (mx + t)$; $r \cosh (mx + t)$ or $r \sinh (mx + t) \pm r \cosh (mx + t)$. Thus, after showing how to pick out the particular pair of functions desired from the family, we may define the trigonometric or hyperbolic functions by adjoining III (*a*) or III (*c*) respectively to I and II. It is interesting to note that as III (*a*), (*b*) and (*c*) are mutually exclusive, the conditions I and II alone define either the trigonometric functions, the hyperbolic functions, or the linear function, the last of which may be considered as a limiting case of either type. This result is natural, in view of the fact that I and the assumption that our functions admit two derivatives would practically restrict them to be solutions of a linear homogeneous differential equation of the second order with constant coefficients. It is interesting, however, to see that the much weaker restriction II is sufficient to give the result stated.

3. *Consequences of* I *and* II. Before proceeding to the separate cases mentioned above, we shall derive from I and II the fact that each member of the family is continuous at all points. To see this, let $G(x)$ and $H(x)$ be the functions mentioned in II, and let them be continuous at x_1. Then

$$G(x + x_2 - x_1) = A_1 G(x) + B_1 H(x),$$

since it is a member of the family by I, where x_2 is any number and A_1 and B_1 are constants depending on $x_2 - x_1$. But since the right member of this equation is continuous at $x = x_1$, the left member is continuous there, and $G(x)$ is continuous

at x_2, any point. Similarly $H(x)$, and therefore every member of the family, is continuous everywhere.

4. *Consequences of* I, II, III (a). We now assume I, II and III (a). Let $F(x)$ be some member of the family (not identically zero) and let $F(b)$ be different from zero. We set

$$\frac{F(b+x)+F(b-x)}{2F(b)} = C(x). \tag{1}$$

It follows from this that $C(x) = C(-x)$; and that $C(0) = 1$. $C(x)$ vanishes by III (a), and as it is continuous, its zeros form a closed set. Let p be the smallest positive value of x for which $C(x)$ vanishes; then $C(p) = C(-p) = 0$ and we set:

$$C(x-p) = S(x). \tag{2}$$

From this we see that $S(0) = 0$; $S(p) = 1$; $S(2p) = 0$. Since $S(x)$ and $C(x)$ are not zero simultaneously, they are independent functions, and therefore every member of the family is a linear combination of them. But, by I, $S(x+h)$ and $C(x+h)$ are members of the family, and we may set

$$S(x+h) = AS(x) + BC(x), \tag{3}$$

$$C(x+h) = A'S(x) + B'C(x). \tag{4}$$

Putting in succession $x = 0$, $x = p$ in (3), and taking account of the particular values previously found, we obtain:

$$S(h) = B, \qquad S(h+p) = C(h) = A, \tag{5}$$

and (3) becomes:

$$S(x+h) = S(x)C(h) + S(h)C(x). \tag{6}$$

We may use this to obtain the value of $S(-p)$. For if we put $h = -x$, we obtain:

$$0 = S(x-x) = S(x)C(-x) + S(-x)C(x),$$
$$0 = C(x)[S(x) + S(-x)].$$

Consequently, $S(x) = -S(-x)$ for all values of x such that $C(x) \neq 0$, and hence, by the definition of p, for all values of x such that $-p < x < p$. But, since $S(x)$ is continuous at all points, we have also

$$S(-p) = -S(p) = -1.$$

This enables us to evaluate the coefficients in (4) by putting

in succession $x = 0$, $x = -p$. This gives

(7) $\qquad C(h) = B', \qquad C(h - p) = S(h) = -A',$

and (4) becomes

(8) $\qquad C(x + h) = C(x)C(h) - S(x)S(h).$

If we put $h = -x$ in (8), we find (at least for values of x numerically less than p)

(9) $\qquad 1 = C^2(x) + S^2(x).$

Consider now a value of x between 0 and p. For such a value, we know from the continuity of the functions and the definition of p that $S(x/2)$ and $C(x/2)$ are both positive. Consequently, by combining (8) (for $h = x$) and (9), we may obtain the formulas:

$$(10) \quad S(x/2) = \sqrt{\frac{1 - C(x)}{2}}, \qquad C(x/2) = \sqrt{\frac{1 + C(x)}{2}}.$$

Combining these relations with (6) and (8), and the values obtained above, $S(0) = 0$, $S(p) = 1$, $C(0) = 1$, $C(p) = 0$, we may calculate by a finite number of operations the values of $S(x)$ and $C(x)$ for any value of x, which divided by p gives a proper fraction expressible as a terminating decimal in the binary scale. Again, since the relations and values used are all satisfied by $S(x) = \sin(\pi x/2p)$, $C(x) = \cos(\pi x/2p)$, the values so obtained will agree with the values for these functions. But, since $S(x)$ and $C(x)$ are continuous, and are equal to these functions for a set of values everywhere dense in the interval $(0, p)$, they must be identical with these functions throughout the entire interval.

Finally, by using the relations (6) and (8), and taking for h, p or $-p$ (recalling that $C(-p) = 0$, $S(-p) = -1$), we may show that if the functions given above represent $S(x)$ and $C(x)$ in any interval, they represent them in an interval longer by p in each direction, and hence for all values of x. This justifies our contention that under assumptions I, II and III (a) the family $AG(x) + BH(x)$ is of the form $A \sin mx + B \cos mx$. Furthermore, if the family is given, we may determine $S(x)$ and $C(x)$ as above, as well as p, and then define $\sin x$ and $\cos x$ by the relations: $\sin x = S(2px/\pi)$; $\cos x = C(2px/\pi)$.

5. *Consequences of* I, II, III (*b*). We next deduce the consequences of I, II, III (*b*). If $F(x)$ is the non-vanishing solution given by III (*b*), $F(x+c)$ and $F(-x)$, which are members of the family by I and evidently never vanish, must be linearly dependent on $F(x)$, giving:

$$F(x+c) = kF(x); \quad F(-x) = jF(x). \tag{11}$$

As the second equation reduces to $F(0) = jF(0)$ for $x = 0$, we see that $j = 1$. Combining this result with the first, we have, taking $c = (x_1 + x_2)/2$,

$$\begin{aligned} F(x_1) = F\left(\frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}\right) &= kF\left(\frac{x_1-x_2}{2}\right) \\ &= kF\left(\frac{x_2-x_1}{2}\right) \\ = F\left(\frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}\right) &= F(x_2). \end{aligned} \tag{12}$$

Since x_1 and x_2 are arbitrary, this shows that the non-vanishing solution $F(x)$ is a constant, and we may take the function $F(x)$ equal to 1.

If $E(x)$ is some non-constant member of the family, it must take on both positive and negative values, since otherwise by adding or subtracting a constant, we should obtain a second member of our family which never vanishes. Hence it must vanish at some point in such a way that the function is positive for all values of the variable in some left (or right) neighborhood of this point, and that in the right (or left) neighborhood of the same length this is not the case. Let b be the value of x at this point, and set

$$P(x) = E(b+x), \tag{13}$$

so that $P(0) = 0$; and note that all members of the family may be expressed in the form $A_1P(x) + B_1$. Then, in particular,

$$P(-x) = jP(x) + k; \tag{14}$$

and, on putting $x = 0$, we see that k is zero, and $P(-x) = jP(x)$. Since

$$P(x) = jP(-x) = j^2P(x), \tag{15}$$

we see that $j = +1$ or -1, and it cannot be $+1$, from the

way in which b was selected, for this value used with (13) and (15) would give

$$E(b + h) = E(b - h),$$

and hence the behavior of $E(x)$ would be the same on the right and left neighborhoods of b. Hence $j = -1$, and

$$P(x) = -P(-x). \tag{16}$$

To obtain the addition theorem for $P(x)$, we notice from I that

$$P(x + h) = AP(x) + B. \tag{17}$$

On taking in succession $x = 0$, and $x = -h$, using (16) in the latter case, we obtain:

$$P(h) = B, \qquad 0 = P(0) = -AP(h) + B. \tag{18}$$

These relations show that $A = 1$, and reduce (17) to

$$P(x + h) = P(x) + P(h). \tag{19}$$

If we set $P(1) = m$, we see from (19) and (16) that, for rational values of x, the values of $P(x)$ equal those of mx. Hence, from the continuity of the function, we have $P(x) = mx$ for all values. Therefore, we have proved that the family is $A + Bx$ in this case.

6. *Consequences of* I, II, III (c). Finally let us consider the consequences of I, II and III (c). Let $F(x)$ be a non-vanishing member of the family, and let c be a number such that

$$F(x + c) \neq kF(-x + c). \tag{20}$$

There must be some choice of the function $F(x)$ and c, for if (20) were not true for some value of c, the reasoning used at the beginning of the discussion of the last case would show that $F(x)$ reduced to a constant, and if this held good for all the non-vanishing solutions, they would all be linearly dependent, and III (c) would be violated. We set

$$\begin{aligned} \frac{F(x + c) + F(-x + c)}{2F(c)} &= C(x), \\ \frac{F(x + c) - F(-x + c)}{K} &= S(x), \end{aligned} \tag{21}$$

where K is a constant to be determined later. It follows

from the definitions that

$$(22)\qquad \begin{aligned} C(0) &= 1, & S(0) &= 0, \\ C(x) &= C(-x), & S(x) &= -S(-x). \end{aligned}$$

As before, we obtain the addition theorems by noting that

$$(23)\qquad S(x+h) = AS(x) + BC(x),$$

$$(24)\qquad C(x+h) = A'S(x) + B'C(x),$$

and evaluating the coefficients. On putting $x = 0$ in (23), we find, by using (22), that

$$(25)\qquad S(h) = B;$$

while, on putting $x = -h$, we find

$$(26)\qquad 0 = S(h-h) = -AS(h) + BC(h).$$

The last two equations show that

$$(27)\qquad 0 = S(h)[C(h) - A],$$

and hence that $A = C(h)$ when $S(h) \neq 0$. Thus, with this restriction, (23) becomes

$$(28)\qquad S(x+h) = S(x)C(h) + S(h)C(x).$$

We remove the restriction by noting that if $S(h) = 0$, while $S(x)$ is not 0, we may interchange the roles of x and h; finally, if $S(x) = S(h) = 0$, (23) and (25), which hold in all cases, show that $S(x+h) = 0$, and thus (28) holds good in this case.

If we put $x = 0$ in (24), we find $C(h) = B'$, and (24) becomes

$$(29)\qquad C(x+h) = A'S(x) + C(x)C(h).$$

If we observe that A' is a function of h, interchange x and h and subtract, we find

$$(30)\qquad A'(h)S(x) = A'(x)S(h).$$

This is an identity in x and h. If we select a value of x, x_1 for which $S(x_1) \neq 0$, and put $A'(x_1)/S(x_1) = k$, we see that $A'(h) = kS(h)$, and (29) becomes

$$(31)\qquad C(x+h) = C(x)C(h) + kS(x)S(h).$$

The number k is positive, negative, or zero. The last case cannot occur, since then we should have

$$(32)\quad C(x+h) = C(x)C(h),\ 1 = C(h-h) = C^2(h),\ C(h) = 1.$$

But this would show that $C(h)$ was a constant, and would reduce (28) to (19), which would prove the family to be

$A + Bx$. This would not satisfy III (e). Also k cannot be negative ($= -k'$), for then we should have

$$C(x+h) = C(x)C(h) - k'S(x)S(h), \tag{33}$$
$$1 = C(h-h) = C^2(h) + k'S^2(h).$$

The second equation shows that for a value of h for which $S(h) \neq 0$, $C(h)$ is less than 1. But $C(h)$ is always positive, since it never vanishes. Since it is continuous, it takes on all values between this value and unity. Since $\cos(\pi/2^n)$ is nearer unity than any fixed number for some value of n, we may find an n and an h_1 such that

$$C(h_1) = \cos(\pi/2^n), \tag{34}$$

and since, as a consequence of (33), we have

$$C(2h) = 2C^2(h) - 1, \tag{35}$$

which is a relation satisfied by the cosine, we see that

$$C(2^{n-1}h_1) = \cos(\pi/2) = 0, \tag{36}$$

which shows that $C(h)$ is not a non-vanishing solution, and hence that the assumption that k is negative leads to a contradiction.

Since k is positive, and becomes k/m^2 if the factor K in the definition of $S(x)$ is replaced by K/m, it may be made unity by a suitable choice of K, and (31) becomes

$$C(x+h) = C(x)C(h) + S(x)S(h). \tag{37}$$

For $h = -x$, this becomes

$$1 = C(x-x) = C^2(x) - S^2(x), \tag{38}$$

while for $h = x$, we get, using (38),

$$C(2x) = 2C^2(x) - 1, \qquad C(x/2) = \sqrt{\frac{1+C(x)}{2}}, \tag{39}$$

where the sign is determined by the fact that $C(x)$ is always positive. Let $C(x_1)$ be a value of $C(x)$ different from unity, and hence greater than unity by (38). Then $C(x_1) = \cosh mx_1$, for some positive m, since $\cosh x$ takes on all values greater than 1 for positive values of x. Hence we see from

$$C(0) = \cosh 0, \qquad C(x_1) = \cosh mx_1, \tag{40}$$

and from the fact that (39) is true for the function $\cosh mx$, that this function has the same values as $C(x)$ for all values

of the form $2^n x_1$ or $x_1/2^n$. Also for these values, we see from (38) that $S(x) = j \sinh mx$, where j is plus or minus one, and must be the same for all such values of x, since (28) shows that

$$S(2x) = 2S(x)C(x). \tag{41}$$

Finally, since (28) and (37) are the addition formulas for $\sinh mx$ and $\cosh mx$, we see that these functions agree with $S(x)$ (to within a sign) and $C(x)$ for all multiples of x_1 whose fractional parts are terminating decimals in the binary scale, and hence, since all the functions concerned are continuous, at all points. Thus under I, II and III (c), the family is necessarily $A \sinh mx + B \cosh mx$.

7. *Conclusions.* In conclusion, we notice that since I and II alone must determine one of the three types of families discussed, we may use any characteristic property of the types in place of III. Thus, we might replace III (a) by the assumption "Some member of the family vanishes twice," or "Every member of the family is bounded." This last statement may be extended so as to give an alternative form of the assumption III, in terms of bounded, instead of non-vanishing functions. That is, III (a), (b) and (c) above may be replaced by the following postulates:

III (a'). There exist two linearly independent members of the family which are bounded.

III (b'). There exists one member of the family which is bounded, and all other bounded members of the family are linearly dependent on this one.

III (c'). No member of the family is bounded.

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GROUPS IN WHICH THE NUMBER OF OPERATORS IN A SET OF CONJUGATES IS EQUAL TO THE ORDER OF THE COMMUTATOR SUBGROUP*

BY G. A. MILLER

1. *Introduction.* From the fact that the commutator quotient group is abelian, it results directly that there is no

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group in which the number of operators in a complete set of conjugates is larger than the order of the commutator subgroup. Hence the groups which satisfy the condition noted in the heading of this paper contain the largest possible number of operators in a complete set of conjugates. They evidently include all the abelian groups as a special case, since the commutator subgroup of every abelian group is the identity. On the other hand, they do not include any simple group of composite order, since such a group is identical with its commutator subgroup.

Each of the groups which satisfy the condition noted in the heading of this paper must involve a co-set as regards its commutator subgroup such that all the operators of this co-set have the same order. Moreover, whenever the commutator subgroup contains more than one Sylow subgroup of a given order, it must transform these Sylow subgroups according to a simply transitive group. If this were not the case, all the operators in a co-set with respect to the commutator subgroup could not transform these Sylow subgroups according to substitutions of the same degree. This common degree would clearly have to be equal to the number of the Sylow subgroups of the same order diminished by one for every set of conjugate Sylow subgroups found in the commutator subgroup.

From the preceding paragraph it results that if a group G contains a set of conjugate operators whose number is equal to the order of the commutator subgroup of G, then this subgroup must satisfy certain conditions. In fact, some groups cannot be the commutator subgroup of any group whatsoever. For instance, it is easy to prove that no dihedral group whose order exceeds 4 can be a commutator subgroup. In fact, none of the non-invariant operators of order 2 contained in such a group could be a commutator since the product of such an operator and some other operator could not transform the generators of the characteristic cyclic subgroup into the same power of themselves as the latter operator. It is also easy to prove that no symmetric group whose order exceeds 2 can be the commutator subgroup of a group.

In fact, if a symmetric group were the commutator subgroup of a group, an operator t outside of this symmetric group would have to be transformed into itself multiplied by an operator represented by a negative substitution of the symmetric group. It is evident that this product could not correspond to t in an automorphism of the group when the order of the symmetric group is neither 2 nor 720. In the special case when the order of the symmetric group is 720 this group admits 720 outer isomorphisms. If t transformed the symmetric group according to an outer isomorphism it results again that the given product could not correspond to t in an automorphism since this product would transform the operators of order 5 in the symmetric group into a different power than t does. While no symmetric group whose order exceeds 2 can be the commutator subgroup of any group whatever, every alternating group is the commutator subgroup of the corresponding symmetric group. An alternating group whose order exceeds 3 cannot, however, be the commutator subgroup of a group in which the number of operators in a set of conjugates is equal to the order of the commutator subgroup. It is interesting to inquire what groups can be commutator subgroups of such groups. In the next section we shall prove that every abelian group has this property.

2. *Abelian Commutator Subgroups.* Let H be an arbitrary abelian group of order h. It is not difficult to construct a group G which has H for its commutator subgroup and involves a set of h conjugate operators. When h is odd such a G can evidently be found by adjoining to H an operator of order 2 which transforms every operator of H into its inverse. The group thus obtained is clearly the generalized dihedral group of order $2h$.

The generalized dihedral group obtained by extending H by means of an operator of order 2 which transforms every operator of H into its inverse has evidently always for its commutator subgroup the group generated by the squares of the operators found in H, and the number of operators in each of its complete sets of conjugate non-invariant operators

of order 2 is equal to the order of this commutator subgroup. When h is even we can therefore construct the required G by first extending H by operators which are commutative with every operator of H and have for their squares the various operators in a set of independent generators of H whose orders are of the form 2^α. By extending the abelian group H thus obtained by means of an operator of order 2 which transforms each of its operators into its inverse we obtain a group which has H for its commutator subgroup and in which the number of operators in the complete sets of conjugate non-invariant operators of order 2 is equal to h.

The order of the G thus constructed is equal to the order of H multiplied by $2^{\beta+1}$, where β is equal to the number of the invariants of H which are of the form 2^α. All the operators whose orders exceed 2 in this G are conjugate in pairs, while the number of the complete sets of h conjugates is 2^β. It is easy to construct by other methods groups which satisfy the given conditions. For instance, if we extend the generalized dihedral group of order $2h$, obtained by extending H by means of an operator of order 2 which transforms every operator of H into its inverse, by means of the operators in its group of isomorphisms which are commutative with every operator of H, there results a group of order $2h^2$ which satisfies the conditions in question.

3. *Number of Conjugates Equal to the Order of the Commutator Subgroup Diminished by One.* The general problem of determining all the groups which satisfy the conditions expressed in the heading of the present paper seems to be quite difficult. There are, however, various special cases which may possibly contribute towards this general solution and which lead to interesting results. Among these is the case when the group G contains a set of conjugate operators whose number is one less than the order k of its commutator subgroup K, in addition to containing a set whose number is equal to k. It is easy to see that the former condition implies the latter while the converse is evidently not true.

In fact, if G involves a set of $k-1$ conjugates, the remaining

operator in the co-set with respect to K to which these conjugates belong must be an invariant operator. Hence K must involve $k-1$ conjugates under G and it must therefore be an abelian prime power group of type $(1, 1, 1, \cdots)$. Since the operators of K are transformed according to a transitive substitution group of degree $k-1$ under G, there must be an operator of G which transforms them according to a substitution of degree $k-1$. As this operator is commutative with only the identity in K it has k conjugates under G. Hence the following theorem has been established:

THEOREM. *If a group contains a set of conjugate operators whose number is equal to the order of the commutator subgroup diminished by one, it must also contain a set of conjugate operators whose number is equal to the order of its commutator subgroup, and this subgroup must be an abelian prime power group of type* $(1, 1, 1, \cdots)$.

No co-set of G with respect to K involves more than one invariant operator, and when it involves one such its remaining operators constitute a single set of conjugates under G. The invariant operators in all of these co-sets constitute the central of G and this has only the identity in common with K. Hence G involves the abelian group which is the direct product of K and the central of G. Each of the operators of G which does not appear in this direct product has k conjugates under G, and the order of G is equal to k times the order of this direct product since the $k-1$ operators of prime order contained in K are transformed under G according to a regular group. The fact that this group is regular results directly from the facts that it is transitive and abelian.

Whenever a group contains a commutator subgroup of prime order p it must also contain at least one set of p conjugates, but when the commutator subgroup is neither of prime order nor the identity the number of operators in a set of conjugates is not necessarily equal to the order of the commutator subgroup. If a group whose commutator subgroup is of order p contains also sets of conjugates involving more than one but less than p operators, each of its co-sets with

respect to K which involves less than p conjugates involves just one invariant operator. The other operators of such a co-set are conjugate in sets of k, where k is a divisor of $p-1$ and has the same value for all such sets.

The fact that the commutator subgroup of G must be abelian whenever G contains a set of conjugate operators whose number is one less than the order of its commutator subgroup may also be regarded as a special case of the theorem that in any group whatever, the difference between the order of the commutator subgroup and the number of operators in a complete set of conjugates is a multiple of the number of the conjugates of any one of the operators of this set under the commutator subgroup. In particular, when there is a set of $k-1$ conjugate operators they must be commutative with every operator of K and hence K must be abelian.

4. *The Number of Conjugates of Every Non-Invariant Operator is Equal to the Order of the Commutator Subgroup.* When the number of operators in every complete set of non-invariant conjugates of the group G is equal to k, the order of its commutator subgroup K, it follows directly that K must be in the central of G since none of its operators could have k conjugates under G. To prove that K must be a prime power group it is only necessary to note that if s is any non-invariant operator of G then the operators of G which are commutative with s constitute an invariant subgroup of index k under G, and the corresponding quotient group is simply isomorphic with K since s transforms the operators of G into themselves multiplied by the various operators of K.

If the order of K is not of the form p^m, p being a prime number, let q and p be two different prime divisors of k. To an operator of order q in the given quotient group there must correspond an operator of order q^β in G. As this operator would have k conjugates under G it would have to be transformed under G into itself multiplied by an operator of order p contained in the central of G. This is evidently impossible since the order of this product could not be of the form q^β Hence we have established the following theorem:

THEOREM. *If the number of operators in every set of non-invariant conjugates of a group is equal to the order of the commutator subgroup, then the order of this subgroup is of the form p^m, p being a prime number.*

From the preceding theorem it results directly that G is the direct product of an abelian group and a non-abelian group whose order is of the form p^m. It is easy to prove that K must be of type $(1, 1, 1, \cdots)$. If this were not the case the operator s of G which was defined above would transform an operator t_1 into itself multiplied by an operator of order p^α while it would transform t_2 into itself multiplied by an operator of order $p^{\alpha+1}$ whose pth power is the multiplier of t_1 under s, $p^{\alpha+1}$ being an operator of highest order contained in K. Hence it may be assumed that $t_2^p = t_1$. This is impossible since G would also have to contain an operator which would transform t_1 into itself multiplied by an operator of order $p^{\alpha+1}$. This proves also incidentally that the pth power of every operator of G is found in the central of G.

In the special case when $k = p$ it is easy to prove that the order of the central quotient group of the Sylow group P of order p^m would have for its order an even power of p. In fact, all the operators which are commutative with a non-invariant operator s_1 of P constitute a subgroup of index p. If s_2 is any operator in P which is not commutative with s_1, then the cross-cut of the subgroups composed of all the operators which are commutative with s_1 and s_2 respectively will be of index p^2 under P. This cross-cut includes the central of P and if it includes other operators we may find a subgroup of index p^2 contained in it and involving the central of P. As this process may be continued until we arrive at the central the theorem in question has been proved.*

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* Cf. J. W. A. Young, AMERICAN JOURNAL, vol. 15 (1893), p. 171.

ON CURVES KINEMATICALLY RELATED TO A GIVEN CURVE*

BY H. PORITSKY

In a space S is given a curve C. With any point P on it is associated a position of the moving trihedron formed by the tangent, principal normal, and binormal to the curve at P. We might call an *indicatrix trihedron* a trihedron whose axes pass through a fixed point, and are parallel to those of the moving trihedron, and denote it by I. As the point P describes the curve, the indicatrix trihedron will rotate about (lines passing through) its vertex. This motion, or the motion of a space rigidly connected to the indicatrix trihedron, with reference to the space S, we shall denote by $[I, S]$. The problem solved in this paper is, to find all the curves C' such that their indicatrix trihedra I' (when drawn with the vertex as I) will experience a motion $[I', S]$ identical with the motion $[I, S]$; in other words, to find curves C', whose points P' can be made to correspond to P so that, as the curves are described by corresponding points, their indicatrix trihedra I, I' remain relatively invariant. Certain interesting families of curves C' are shown to exist. A kinematic method of treatment has been adopted.

Denote unit vectors along the tangent, principal normal, and binormal of C by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$,† respectively. It is known that the motion $[I, S]$ is completely characterized by the fact that there is no component of rotation along the principal normal $\boldsymbol{j}$.‡ The components of the rotation along $\boldsymbol{i}$, $\boldsymbol{k}$, if P describes C with unit velocity, may be identified with the torsion τ and the curvature κ. If the velocity of P is not unity but $v = ds/dt$ (s is arc length, t time), the components of rotations become $v\tau$, $v\kappa$. Thus the rotation vector always lies in the plane π, through $\boldsymbol{i}$, $\boldsymbol{k}$, and parallel to the rectifying plane. It will be noticed that π is a plane of (i.e., fixed relative to) the moving space I'.

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† Clarendon type will be used to denote vectors.

‡ Darboux, *Théorie des Surfaces*, 2d ed., vol. I, p. 13.

We must now consider two cases:

1. The motion $[I, S]$ consists of a rotation about a fixed line.

2. The motion $[I, S]$ may be generated by rolling a cone fixed in I upon a cone fixed in S. The cones are the loci of instantaneous axes of revolution in I and S respectively. Case 1 is the special instance when the cones degenerate into lines.

Since, as just mentioned, the rotation vector has no component along $\boldsymbol{j}$, the rolling cone, whose elements bear the rotation vectors, reduces to the plane π (case 2), or else the fixed axis of rotation is a line in π (case 1). In the latter case the components of rotation along $\boldsymbol{i}$, $\boldsymbol{k}$ have a fixed ratio. Hence

$$\frac{\kappa}{\tau} = \text{const.} = \tan \alpha,$$

where α is the angle which the fixed rotation axis makes with $\boldsymbol{i}$. C is known to be a cylindrical helix. Leaving this case for later consideration, we shall first consider the general case.

Let primed letters indicate objects referring to C' of the same kind as the unprimed letters denote for C. Let corresponding points on the two curves be given by the same value of t; thus $P(t)$, $P'(t)$ will denote corresponding points on C, C', while P, P' denote merely any two points on C, C'; similarly $\boldsymbol{i}(t)$, $\boldsymbol{i}'(t)$ will be used for unit tangents at two corresponding points, and so forth.

If the motions $[I, S]$, $[I', S]$ be identical, the fixed and the rolling cones for the two motions, as well as the angular velocities, must be the same. But since the rolling cones reduce to the planes π, π' which are parallel to the rectifying planes at $P(t)$, $P'(t)$, respectively, $\pi(t)$, $\pi'(t)$ must coincide, and so must the principal normals $\boldsymbol{j}(t)$, $\boldsymbol{j}'(t)$ which are perpendicular to π, π'. The rotation by means of which $I(t)$ is transformed into $I'(t)$ ($\boldsymbol{i}(t), \boldsymbol{j}(t), \boldsymbol{k}(t)$ going into $\boldsymbol{i}'(t), \boldsymbol{j}'(t), \boldsymbol{k}'(t)$) is therefore about the line bearing $\boldsymbol{j}(t), \boldsymbol{j}'(t)$, and is of a constant angle θ, since the two indicatrix trihedra are relatively fixed. Resolving the unit vectors of I' along those of I, we get

$$\begin{aligned} \boldsymbol{i}'(t) &= \boldsymbol{i}(t)\cos\theta + \boldsymbol{k}(t)\sin\theta, \\ \boldsymbol{j}'(t) &= \boldsymbol{j}(t), \\ \boldsymbol{k}'(t) &= -\boldsymbol{i}(t)\sin\theta + \boldsymbol{k}(t)\cos\theta, \end{aligned} \tag{1}$$

θ being independent of t.

Equating the rotation vectors

$$v'(t)[\tau'(t)\boldsymbol{i}'(t) + \kappa'(t)\boldsymbol{k}'(t)] = v(t)[\tau(t)\boldsymbol{i}(t) + \kappa(t)\boldsymbol{k}(t)], \tag{2}$$

we have identified the fixed cones and the angular velocities. Solving (1) for $\boldsymbol{i}$, $\boldsymbol{k}$, in terms of $\boldsymbol{i}'$, $\boldsymbol{k}'$, substituting in (2) and equating coefficients of like vectors we get

$$\begin{aligned} v'(t)\tau'(t) &= v(t)\tau(t)\cos\theta + v(t)\kappa(t)\sin\theta, \\ v'(t)\kappa'(t) &= -v(t)\tau(t)\sin\theta + v(t)\kappa(t)\cos\theta. \end{aligned} \tag{3}$$

Nothing is lost in generality if we identify the arc length along the curve C with time, thus letting $v(t) = 1$.

Having chosen θ, the successive positions of I' and consequently also of $\boldsymbol{i}'$ are determined as functions of t. The curve C' itself is not specified till we give $v' = ds/dt$, whereupon we obtain C' by vectorial integration of $v'(t)\boldsymbol{i}'(t)$ (one must also give the starting point). All the curves that we obtain from a fixed θ by varying v' can be made to correspond to each other, so that at corresponding points (given by the same values of t), the indicatrix trihedra coincide, and are thus to each other in relation of Combescure.

We shall restrict ourselves for the present to the case where $v' = 1$, but will let θ take on any value between 0 and 2π; the curves C', which we are seeking, will have any two of their points the same arc length apart as the points of C to which they correspond. We may now put $s' = s = t$ where s' is the arc length along C; (3) reduces to

$$\begin{aligned} \tau'(t) &= \tau(t)\cos\theta + \kappa(t)\sin\theta, \\ \kappa'(t) &= -\tau(t)\sin\theta + \kappa(t)\cos\theta. \end{aligned} \tag{3'}$$

If we start all the curves C' with the point $s' = 0$ at a common starting point Q, there will be obtained a one-parameter family of curves F, one curve corresponding to every value of the parameter θ. The given curve C corresponds to $\theta = 0$. All other curves that have an indicatrix trihedron I' fixed relative to I may be obtained from some one curve of F by a transformation of Combescure. (3') shows

that the curvatures and torsions of the curves of F are certain linear combinations of those of the initial curve. The family F has some interesting properties which we now proceed to investigate.

Denote by $\boldsymbol{P}(\theta, s)$ the vector drawn from the point Q to any point P' on the curves F; similarly replace $\boldsymbol{i}'(t)$, $\boldsymbol{j}'(t)$, $\boldsymbol{k}'(t)$ by $\boldsymbol{i}(\theta, s)$, $\boldsymbol{j}(\theta, s)$, $\boldsymbol{k}(\theta, s)$. Between $\boldsymbol{P}$ and $\boldsymbol{i}$ we have the relation

$$\frac{\partial \boldsymbol{P}(\theta, s)}{\partial s} = \boldsymbol{i}(\theta, s), \tag{4}$$

whence by integration, since $\boldsymbol{P}(\theta, 0) = 0$,

$$\boldsymbol{P}(\theta, s) = \int_0^s \boldsymbol{i}(\theta, s)ds. \tag{4'}$$

Using (1) we obtain

$$\boldsymbol{P}(\theta, s) = \cos\theta \int_0^s \boldsymbol{i}(0, s)ds + \sin\theta \int_0^s \boldsymbol{k}(0, s)ds. \tag{5}$$

Since $\boldsymbol{k}(\theta, s) = \boldsymbol{i}(\theta + \frac{\pi}{2}, s)$, (5) becomes

$$\boldsymbol{P}(\theta, s) = \cos\theta \int_0^s \boldsymbol{i}(0, s)ds + \sin\theta \int_0^s \boldsymbol{i}\left(\frac{\pi}{2}, s\right)ds, \tag{5'}$$

and, using (4'), we find

$$\boldsymbol{P}(\theta, s) = \cos\theta\, \boldsymbol{P}(0, s) + \sin\theta\, \boldsymbol{P}\left(\frac{\pi}{2}, s\right). \tag{6}$$

If we hold s constant and let θ run between 0 and 2π, the end point of $\boldsymbol{P}(\theta, s)$ will describe the locus of the corresponding points on the curves F, which are at an arc length s from Q. From (6) we conclude that these corresponding points always lie on ellipses E. The latter have their centers at Q, and the vectors $\boldsymbol{P}(0, s)$, $\boldsymbol{P}(\pi/2, s)$ are conjugate semi-axes. Obviously, any two curves whose values of θ differ by $\pi/2$ will yield conjugate axes of the ellipses. We might call such curves conjugates of each other. The tangents to two conjugate curves are perpendicular to each other.

When s is small the ellipses approximate a circle. For, from (5), $\boldsymbol{P}(\theta, s) = [\cos\theta\ \boldsymbol{i}(0, 0) + \sin\theta\ \boldsymbol{k}(0, 0)]s$ approximately for small s. The plane of this circle is perpendicular to the common principal normal of F at Q.

In the plane $\pi(0)$ through $\boldsymbol{i}(0, 0)$, $\boldsymbol{k}(0, 0)$, let us set up a

system of rectangular Cartesian axes (x, y) with origin at the vertex of I, the end-points of the two unit vectors mentioned having coordinates (1, 0), (0, 1), respectively. The unit circle described by $\boldsymbol{i}(\theta, 0)$ now has for its parametric equations

$$x = \cos\theta, \qquad y = \sin\theta. \tag{7}$$

Similarly, in the plane of the ellipse E, described by $\boldsymbol{P}(\theta, s)$ when s is held fixed, let us set up a system of oblique Cartesian coordinates (ξ, η) the origin being at Q, while the axes and scales on them are chosen so that the coordinates of $\boldsymbol{P}(0, s)$, $\boldsymbol{P}(\pi/2, s)$ are (1, 0) and (0, 1), respectively. Any vector $a\boldsymbol{P}(0, s) + b\boldsymbol{P}(\pi/2, s)$ will then have for the coordinates of its end-points $\xi = a$, $\eta = b$. Hence, using (6) we get for the parametric equations of the ellipse,

$$\xi = \cos\theta, \qquad \eta = \sin\theta. \tag{8}$$

Now consider any affine projective transformation carrying the plane (x, y) into (ξ, η) so that for corresponding points $x = \xi$ and $y = \eta$. Such an affine projective transformation will send the circle (7) into the ellipse (8), corresponding points being given by the same value of θ. Therefore, corresponding points on two ellipses are projectively related. If we join such points by drawing corresponding secants for the curves, that is, secants between two corresponding points, we obtain a ruled surface of (not higher than) the fourth degree. By passing to the limit we get a similar result for the tangents.

The result obtained that points of F lie on ellipses E is only a special case of a more general theorem. If we call M the motion described by the moving trihedron of F as θ is varied but s is held fixed, and denote by R any point fixed to that trihedron, then the path of R as a result of the motion M will be an ellipse. For

$$\begin{aligned}\vec{QR} &= \vec{QP} + \vec{PR} \\ &= \boldsymbol{P}(\theta, s) + c_1\boldsymbol{i}(\theta, s) + c_2\boldsymbol{j}(\theta, s) + c_3\boldsymbol{k}(\theta, s),\end{aligned}$$

where c_1, c_2, c_3 are constants. Using (1) and (6) we obtain

$$\begin{aligned}\vec{QR} = {} & \cos\theta[\boldsymbol{P}(0, s) + c_1\boldsymbol{i}(0, s) + c_3\boldsymbol{k}(0, s)] \\ & + \sin\theta\left[\boldsymbol{P}\left(\frac{\pi}{2}, s\right) + c_1\boldsymbol{i}\left(\frac{\pi}{2}, s\right) + c_3\boldsymbol{k}\left(\frac{\pi}{2}, s\right)\right] + c_2\boldsymbol{j}(0, s);\end{aligned}$$

s being held fixed $= s_0$, this is the equation of an ellipse having its center at $c_2 j(0, s_0)$, that is, on the line through Q having the direction of the principal normals at the points $P(\theta, s_0)$.

The movements M in which all points describe ellipses are of the type described by Darboux in one of his notes to Koenigs' *Kinematics*.* It is there proved that this is the only possible movement in which the trajectories of all points are plane curves, except for the case where the trajectories lie in a family of parallel planes. A line generally describes a surface of the fourth degree. Such a motion may be generated by proper rolling and slipping of two circular cylinders whose radii are in the ratio 1 : 2. The axes of these cylinders will in the present instance be parallel to the principal normals.

Since the relation (3′) between κ, τ and κ', τ' is the same as between the coordinates of the same point in a plane referred to two sets of rectangular axes with a common origin and forming an angle θ with each other, it follows that if κ, τ satisfy an algebraic relation, κ', τ' will satisfy a relation of the same degree. Thus, if κ, τ satisfy a linear relation so that C is a Bertrand curve, all the curves of F will be Bertrand curves. There will then exist two values of θ for which $\kappa' = \text{const.}$, and two other values differing from the former by $\pi/2$ for which $\tau' = \text{const.}$ The latter curves are the conjugates of the former.

The expression $[\kappa^2 + \tau^2]^{1/2}$ gives the rate at which the principal normal or the rectifying plane turns, and is sometimes spoken of as the "normal" curvature. From (3) it follows that for all the curves of F, the normal curvature is the same. In particular, if it is constant for C, it is equal to the same constant for any curve of F.

F was obtained by putting $v' = ds'/ds$ equal to unity, that is, by letting corresponding points on C and C' be equal arc lengths apart. Other interesting families of curves may, no doubt, be obtained by choosing other proper expressions for v'. Thus, if we let v' be a function of θ only, $v' = v(\theta)$, we can go through a treatment similar to that for the F family. Since

* G. Koenigs, *Leçons de Cinématique*, Note IV, §§ 2, 3, p. 252.

we may no longer use the same symbol s to denote arc length along any curve, we shall think of corresponding points along different curves as given by the same value of t. We now have (in place of (4))

$$\frac{\partial \boldsymbol{P}(\theta, t)}{\partial t} = \frac{\partial \boldsymbol{P}(\theta, t)}{\partial s'} \frac{ds'}{dt} = v(\theta) \cdot \frac{\partial \boldsymbol{P}}{\partial s'} = v(\theta) \boldsymbol{i}(\theta, t);$$

and, instead of (6),

$$\boldsymbol{P}(\theta, t) = \cos\theta\, v(\theta) \boldsymbol{P}(0, t) + \sin\theta\, v(\theta) \boldsymbol{P}\left(\frac{\pi}{2}, t\right). \tag{9}$$

This is the locus of corresponding points on the curves now. It is still a plane curve and may be regarded as an affine projective image of the curve whose equation in polar coordinates is $\rho = v(\theta)$, this curve taking the place of the unit circle of which the ellipses of family F were projective images.

In case $\rho = v(\theta)$ is an algebraic curve, it can be shown that the movements M, described by the moving trihedra of the curves as θ varies while t is fixed, are still algebraic, that is, the path of any point during this motion is an algebraic curve; its degree is considerably higher than that of $\rho = v(\theta)$. Thus if we take $\rho = \sec\theta$, the corresponding points of the curves always lie on straight lines. During the motion M the general point can be shown to describe a curve of the fourth degree which is an affine projective image of

$$z = \tan\theta, \qquad x = \cos\theta, \qquad y = \sin\theta.$$

Now let us return to the case where C is a cylindrical helix, the motion $[I, S]$ reducing to a rotation of I about a fixed axis l (whose direction gives the direction of the rulings of the cylinder). The positions I' could take up relative to I are now of two parameters instead of one. For, if the position of $\boldsymbol{i}$ be arbitrarily assigned, since the rotation vector lying on l must always be in the plane $\boldsymbol{i}'$, $\boldsymbol{k}'$, the position of I' is completely specified (except for the case when $\boldsymbol{i}$ coincides with l). $I'(t)$ is now determined in time, and it remains only to choose $v'(t)$ and the initial point in order to specify C'. Whichever $I'(t)$ we pick out, it rotates about l, $\boldsymbol{i}$ forming a constant angle with it. Hence all the curves C' are cylindrical helices, the

elements of whose cylinders are parallel to those of C. The curves obtained by putting $v' = 1$, and prescribing a common starting point Q form a two-parameter family of curves G, such that any other curve whose indicatrix I' is fixed relative to I is in a relation of Combescure to some curve of G.

If we restrict $\boldsymbol{i}$ to lie in a plane through l, we pick out a one-parameter group of positions of I, which is of the type previously discussed for a general curve C. We may now construct the family F by picking out a position of $\boldsymbol{i}$ to measure θ from. For two values of θ (differing by π) the unit tangent will coincide with l, yielding for the curve C a straight line C_s. The two curves C_p that are conjugate to C_s will have their tangents perpendicular to the line C_s, and consequently lie in a plane perpendicular to C_s. The vectors joining Q to corresponding points on C_s, C_p will form conjugate axes of the ellipse E, and being perpendicular to each other will now be the major and minor axes of E. The points of F can thus be obtained by having an ellipse move so that its major axis increases at a uniform rate along a straight line while the minor axis describes a plane curve with the same velocity.

If we now rotate about l the plane in which we restricted $\boldsymbol{i}$ to lie, all possible positions of I' (relative to I) are obtained. The infinitesimal vectors $\boldsymbol{ids}$ experience a fixed rotation about l, and the same is true of the sum vector $\int_0^s \boldsymbol{ids}$. Hence, all the other curves of G may be obtained from the one-parameter family just considered by rotating the latter about l.

We see, therefore, that if we hold s constant, the locus of a point fixed relative to I' is a surface of the fourth degree obtained by revolving an ellipse about the line l. In particular the locus of the vertex of I' which describes the curves C themselves reduces to an ellipsoid of revolution since the ellipse has its major axis on l. The torsions and curvatures of C' are again linear combinations of those of C, the relation being given by (3′). Hence, if C is a circular helix, so that τ, κ are constant, τ', κ' will also be constant, and all the curves of G will be circular helices.

CORNELL UNIVERSITY

HEATH ON GREEK MATHEMATICS

A History of Greek Mathematics. By Sir Thomas Heath. Volume I, *From Thales to Euclid;* volume II, *From Aristarchus to Diophantus*. Oxford, Clarendon Press, 1921. xvi + 446 + xii + 586 pp.

It is doubtful if anyone could assume the privilege of writing a review (at least an American review) of this monumental treatise without a feeling of helplessness. An American review must, by our present standard of taste in matters literary and critical, be brief, be popular, and (according to certain other standards) be both flippant and faultfinding; but no brief review can do this work justice nor can a sufficiently extended review be popular, while either flippancy or faultfinding would be merely a display of poor taste, like loud boasting or any other of the various species of vulgarity.

With such thoughts, a reviewer may properly ask himself what, precisely, is his mission. Is it to seek out points of doubt in the narrative and magnify them into blunders, or to search out minor inaccuracies (for such can always be found in any book) and display them as typical of the work under inspection? Should we take seriously Montaigne's *mot*, "Since we cannot attain to greatness, let us have our revenge by railing at it"? Or has the critic a more wholesome duty to his readers—that of stating the salient features of an author's work and of discovering whether he has seen things as they really are, has recorded them with becoming felicity of style, and has searched out such of the causes for world progress as can be shown to exist?

In considering the work under review, therefore, it is proposed to ask whether the author has seen Greek mathematics as it was, and whether he has expressed the results of his studies in a style that will command the willing attention of those whose tastes will lead them to read a work of this nature. In the seeing of Greek mathematics as it was, there is involved the question of causes, of the general setting of mathematics in the philosophical schools of that remarkable race, and of the balancing of the merits of different investigators and expositors. In the matter of style, a subject admitting of only slight mention, there may well be raised the question of the influence upon the author of the Greek language in which the science was expressed—a subject not without value in these days in which Greek is reported to have become obsolete as a subject of study in the schools of our country.

First, then, has the author seen the Greek mathematicians as it were face to face, intellect to intellect, as one master of the subject to another? —or has he seen them as in a glass, darkly, trusting to secondary sources and to unexplored tradition? The answer is a simple one—that no man now living is more capable than he of interpreting the Greek mathematical mind to the scholar of today; indeed, there is no one who ranks even in the same class with Sir Thomas Heath in this particular. Paul Tannery

might have done so in his lifetime, but in knowing the mathematics of Greece, in knowing the Greek of mathematics, and in knowing the causes which led to the great development of the science in Miletus, in Crotona, in Athens, in Alexandria, and in the islands of the Ægean Sea, the author of this treatise stands without a rival, and these causes he has set forth for us in his usually lucid style.

This is a strong assertion, it sounds exaggerated, and it has the odor of that type of flattery from which Byron and many another of lesser fame has begged to be delivered—and yet the assertion is perfectly true. Sir Thomas Heath ranked in his university as a leader in mathematics and as a leader in Greek, and thus he was equipped as few others have been to see Greek mathematics with an intimacy that most men have been unable to enjoy, and he has set forth the results of his intimate knowledge with a felicity of expression that could come only from familiarity with the language of the men whose works he has described.

The first difficulty that the historian meets in making known the results of his studies is analogous to the laying out of the ground plan of a structure; it requires him to consider which of several designs he will take and what shall be the order of rooms through which the visitors to the edifice shall be conducted. Moreover, in the case of the history of any science a writer is confronted by one particularly serious difficulty, namely, that of the sequence of chapters. A reader may properly demand that the treatise present the growth of the subject in chronological order, and that it shall also present a subject like analytic geometry as a unit. These two demands are, however, mutually antagonistic, since the chronological order of a history of mathematics would scatter the evidences of the development of any of the leading topics along a period of two or three thousand years, while any mere topical treatment would require a wearisome repetition of chronological and biographical material with every topic considered. Historians are continually trying to harmonize these methods, even as astronomers have tried from time immemorial to harmonize the lunar and solar calendars, and the result in each case is necessarily a compromise. The author has himself called attention to the difficulty, characterizing Professor Loria's *Le Scienze esatte nell' antica Grecia* as "the best history of Greek mathematics which exists at present," and showing that the distinguished Italian historian had taken, as he frankly states, "a compromise between arrangement according to subjects and a strict adherence to chronological order, each of which plans has advantages and disadvantages of its own."

In the work under review the author has made the attempt to solve the problem by arranging his chapters as follows:

I. *Introductory*, the purpose being to give the reader a general view, as from an airplane, of the terrain through which he is to be led—a desirable preliminary treatment in any historical treatise.

II. *Greek numerical notation and arithmetical operations*, setting forth what the reader should know of that part of the numerical art which the Greeks called "logistiké."

III. *Pythagorean arithmetic*, in which is given in considerable detail a survey of that which was known as "arithmetiké"—the theory of numbers.

IV. *The earliest Greek geometry*, in which are set forth the life and labors of Thales. It will be noticed that in Chapters II–IV the author is forced to abandon a strictly chronological sequence, Chapter II considerably overlapping Chapter III, and each overlapping Chapter IV. This is a necessity for a writer who attempts a topical arrangement, and it results in this case in the omission from Chapter III of the contributions of Euclid and Diophantus, for example, these being considered later in chapters devoted to the men themselves.

V. *Pythagorean geometry*, in which the story of geometry is continued, but in which the difficulty is necessarily encountered of distributing the biography of Pythagoras between Chapters III and V. It is interesting to observe that the author has less to say of the life and times of this great philosopher than of any other leading Greek mathematician, probably because less authentic material is available for a biographical sketch. Concerning the much mooted question as to the source of the first demonstration of the theorem which bears the name of Pythagoras, the author gives a judicial summary of the evidence and concludes with the statement: "I would not go so far as to deny to Pythagoras the credit of the discovery of our proposition; nay, I like to believe that tradition is right, and that it was really his"—a decision that will meet with the approval and command the respect of the great majority of students of history.

VI. *Progress in the Elements down to Plato's time*, in which a study is made of one of the most interesting periods in the development of Greek geometry—the formative stage in which proofs were discovered and the logical bases of the science were beginning to be sought. It is now possible for the author to give to the treatise an arrangement that is more nearly biographical, and to set forth the biographies in chronological sequence.

VII. *Special problems*, in which the "three famous problems" of antiquity are considered. Here, as in Chapter VI, the nature of the subject permits of the biographical and chronological treatment. Among other details, Bryson's contributions to the study of the method of exhaustion are recognized more favorably than has of late been the case.

VIII. *Zeno of Elea*. It is probably quite justifiable to give Zeno a chapter by himself, since it would be difficult to place him with anyone else. While the author has not carried his study of the history of the philosophic interpretation of Zeno's problems as far as our Professor Cajori (to whose contribution he pays just tribute), he condenses in a few pages the best of the Greek interpretations of his paradoxes.

IX. *Plato*, in which chapter there is given a succinct statement of the influence of this great philosopher with respect to the foundations upon which a work like Euclid's should rest, this statement being fortified by extracts from Plato's works as well as from those of subsequent writers who were conversant with his doctrine.

X. *From Plato to Euclid*, the period in which the post-Pythagorean accumulation of propositions and the influence of Plato with respect to

foundation principles were working towards the making of a treatise which should set forth the Greek geometry in all its excellence. In this period falls the work of Aristotle, not generally enough appreciated in its influence upon geometry but here given just recognition.

XI. *Euclid*, whom no one in our time has recognized so worthily as Sir Thomas Heath, and to whom has here been given about one fourth of Volume I. Naturally the author has done little more than condense into about a hundred pages his own monumental treatise upon the *Elements*, using much of the language there employed, and he could not have done better than follow this plan. Since the belief is not uncommon that Euclid was merely a textbook writer, devoid of mathematical genius, the tribute here paid, showing his genuine powers as a geometer, is welcome. It has been said of Shakespeare that he "took the stillborn children of lesser men's brains and breathed on them the breath of life," and Euclid may have done the same, but it takes a genius, perhaps a divine genius, to perform this miracle.

XII. *Aristarchus of Samos*, more of an astronomer than a geometer, but nevertheless one of the first great geometricians in the astronomical field. It is characteristic of the author that he makes no mention of his own treatise on Aristarchus, indeed, that he hardly refers to any of his other works. This is an illustration of British modesty, to the lack of what they call "side," which we in America (probably unfortunately) fail to understand; for not infrequently the reader might be assisted by more frequent reference to such standard works as those which the author has contributed to the study of Greek mathematics.

XIII. *Archimedes*. It is a tribute to five of the greatest names in the field of ancient mathematical research and exposition that the author has given to each approximately a hundred pages, the total amounting to about half of the entire treatise. These men are Euclid, Archimedes, Apollonius, Pappus, and Diophantus. Of these it is an easy matter to pick out the least, but it is difficult to select the greatest. Probably, if a ballot were taken among those who have the intellectual right to vote, the choice would fall upon Archimedes, and the treatment which the author has accorded him is in harmony with this judgment.

XIV. *Conic Sections: Apollonius of Perga*. As in the cases of Euclid, Aristarchus, Archimedes, and Diophantus, the intellectual world is already familiar with the author's treatises upon this great expositor (and doubtless largely the creator) of the ancient theory of conics. There is no better way of securing an insight into the essential difference between the mathematics of the Greeks and that of the present day than by comparing the treatment, say of the ellipse, as given by Apollonius, with that given in our own modern textbooks, and perhaps there is nothing that gives a student a higher appreciation of the Greek mind. For one who wishes this opportunity but who has but little time for the comparison, this chapter will prove especially helpful.

XV. *The successors of the great geometers*, in which chapter is given a brief statement of the work of those who began, in any large way, the

theory of higher plane curves—a theory which Greece had no longer the intellectual strength to complete. The names considered include those of Nicomedes, Diocles, and Perseus—each of whom is known chiefly for a single important contribution to geometry.

XVI. *Some handbooks*, under which properly depreciative title there lie the works of Cleomedes, Nicomachus, and Theon of Smyrna. If ever a man was pushed into fame in the field of mathematics, by little more than chance, that man was Nicomachus.

XVII. *Trigonometry: Hipparchus, Menelaus, Ptolemy*, in this order, for "the first person to make systematic use of trigonometry is, so far as we know, Hipparchus." It was with him that the long union of trigonometry and astronomy began, a union particularly noticeable in the Arab schools, and only broken when each science, in the fifteenth century, had so developed that it was able to stand alone and to set up an establishment for itself.

XVIII. *Mensuration: Heron of Alexandria.* Aside from setting forth a summary of Heron's work the author devotes considerable attention to the controversy which has so long been waged as to the time in which this great scholar lived. Not even Diophantus has given historians so much trouble, partly due to the fact that there were a number of Herons whose names have come down to us together with some knowledge of their achievements. Until recently the Heron of mathematical fame has been commonly placed in the first century B.C. (and sometimes earlier), but of late he has been put in the first century of our era. Heiberg felt that the third century would be a safer conjecture, and Sir Thomas Heath, after carefully weighing the evidence, thinks the same. The evidence is by no means conclusive, but it seems certain that the date was about 50 A.D. or not more than two centuries thereafter.

XIX. *Pappus of Alexandria*, the last of the prominent geometers of Greece; not a great genius but, considering the time in which he lived, a great scholar and a worthy mathematician.

XX. *Algebra: Diophantus of Alexandria*, much more of a genius than Pappus or Heron, a great mathematician in a period of general scientific decay, and the one who best deserves the title of "father of algebra."

XXI. *Commentators and Byzantines.* Under this rather interesting caption, with its hint at mutual exclusion, are considered the names and works of men like Serenus, Theon of Alexandria, Proclus, and Psellus, and of the only prominent woman mathematician of all antiquity—Hypatia. It was the period of the death of Greek science, and, as with all such periods, its chronicles are not stimulating to the mind.

The second volume has a good index and a list of Greek terms.

The limits set for reviews at the present time are such as to allow no adequate statement of the merits of this noteworthy treatise. It is only possible to add that it is destined to be the standard work upon the subject, even as the author's other works are the recognized standards in their respective domains.

The feature which, with respect to the substance of the text, distinguishes the work from any other of its kind is the large amount of source material

that it contains. The author has not merely written a history; he has set forth at unusual length the evidence to support his views. Thus we have extracts generously made from such writers as Archimedes, Apollonius, Pappus, and Diophantus by which the reader is able to form a first-hand opinion of the nature and value of their several contributions and methods of attack. Such a plan operates against a literary production of uniform smoothness of expression, and it considers the needs of the scholar instead of the taste of the casual reader; but it is precisely the scholar whom the author has in mind rather than a more general intellectual public for which men like Gibbon and Guizot wrote their justly celebrated treatises. It is difficult to accomplish both purposes; Libri attempted it by giving the scientific world a text composed with true French elegance and supplementing this by source material in an appendix, but in so doing he made the necessary sacrifice of any close connection between his text and his evidence, and moreover his selection of material was rather on the basis of rarity than of importance. On the whole, indeed, no one has solved the problem more effectively than has been done in the work under review.

It should also be said that the plan of quoting so fully from the ancient writers serves to remove any doubt as to the validity of the author's thesis that "the foundations of mathematics and a great portion of its content are Greek. The Greeks laid down the first principles, invented the methods *ab initio*, and fixed the terminology. Mathematics in short is a Greek science, whatever new developments modern analysis has brought or may bring." To establish such a thesis there is demanded not merely the assertions of today; we must have the precise evidence of the past, and this is what has been placed before the reader in such abundance as to make the work a source-book as well as a historical narrative.

If it should be asserted, as the reviewer has already heard it remarked, that the arrangement of material suggests a set of essays rather than a connected historical discourse, the obvious reply is that the history of Greek mathematics is largely a record of the work of a few great geniuses; that it is not, like political history, a list of innumerable wars and changes of dynasties and of perpetual slaughter and rapine, nor even like economic or social history, and that it therefore requires precisely the treatment here accorded to the story of the leaders of scientific thought. It is idle to speculate as to whether the preliminary historical essay might have been expanded into a volume, and the biographical material have been there disposed of first, the special topics being placed in the second volume. Suffice it to say that the author felt that he could accomplish his purpose better by pursuing the route he has taken, and every writer should take the road he can travel most safely. The goal has been reached, the way has been delightful, the guide has spoken as one having authority, and the scientific world should be accordingly generous in its praise and in its appreciation and its thanks.

DAVID EUGENE SMITH.

SHORTER NOTICES

Lezione di Geometria Proiettiva. Federico Amodeo. Third edition, second reprint. Naples, Luigi Pierro, 1920. 450 pp.

The second reprint of the third edition of this work contains an appendix (43 pp.) of modifications and additions made since the first appearance of the third edition (1905).

The text consists of two parts and each part of three chapters. There is an introductory chapter (66 pp.) preceding Part I in which is set forth the fundamental postulates: definitions of primitive forms; the notions of perspectivity, projectivity, homography, correlation, etc.; the theorem of Desargues and its consequences.

The fundamental postulates are those formulated by Amodeo in ATTI DELL'ACCADEMIA DELLE SCIENZE DI TORINO, March, 1891. The first five deal with the existence of elements, denoted by S_i, $i=0, 1, 2, 3$. The sixth postulate asserts the closure of S_1, considered as a class of S_0's, and thus provides for an ideal point on each line (when $S_0 \equiv$ point, $S_1 \equiv$ line). Postulate seven asserts the invariance of the property of separation of two couples of elements on a primitive form under projection and section. There are two more postulates contained in Chapter I of Part I. The first is equivalent to the Postulate of Archimedes from which is deduced the existence of an infinity of elements of a "succession" (previously defined by harmonic groups) between any two elements of the succession; and the second is the Dedekind Postulate from which is deduced the continuity of each primitive form.

Part I, 217 pages, is devoted to the projective geometry of "simple forms" (primitive forms). It contains Von Staudt's Theorem and its consequences; the theory of projectivities upon primitive forms, of involutions, of polarities in S_2 and S_3; the definition of imaginary elements; etc. Polarities in euclidean space (S_2 and S_3) have centers, diameters, axes, foci, but no explicit mention is here made of conics or cones.

Imaginary elements on any primitive form are defined as double points of an elliptic projectivity upon the form, and are represented by four elements $A\ B\ C\ D$ such that A corresponds to B, B to C, and C to D ($A\ D$ not separated by $B\ C$). This method was given by Amodeo in lectures (lithographed), 1887–88, on homographies in the binary field, and printed in GIORNALE DI BATTAGLINI, volume 26, 1888, in a paper entitled *Fasci di omographie e rappresentazione geometrica degli elementi imaginarii.* Professor J. Rey Pastor, in his book *Fundamentos de la Geometria Proyectiva Superior* (1916), calls this "the method of Amodeo." But the method apparently differs very little from that of Von Staudt, since a pair of conjugate imaginary elements defined as above is also defined by a unique elliptic involution; namely the double-points involution (Involuzione unita, Doppelpunktsinvolution).

Part II, 165 pp., is devoted to forms of the second order and one dimension. These forms are defined by means of projectively related

primitive forms rather than as loci of corresponding incident elements in polarities, or as loci corresponding to circles in perspectively related forms, thus following the road chosen by Reye rather than that selected by Von Staudt, or Steiner and Cremona.

The book as a whole is full of detail with good sets of exercises after each chapter. The presentation is clear. The method is synthetic for the most part, although the author does not hesitate to use projective coordinates and the resulting analysis somewhat freely throughout the book. For students who wish to pursue geometry beyond the ordinary undergraduate curriculum in our American universities, Professor Amodeo's book will be both suggestive and inspiring.

L. Wayland Dowling

United States Life Tables 1890, 1901, 1910 and 1901–10. Explanatory Text, Mathematical Theory, Computations, Graphs, and Original Statistics. Also Tables of Life Annuities, Life Tables of Foreign Countries, Mortality Tables of Life Insurance Companies. By James W. Glover. Washington, Government Printing Office, 1921. 496 pp.

The preparation of this volume marks an important advance in the study of vital statistics in the United States.

The work is divided into eight parts. Part I gives a nontechnical description and explanation of life table functions. Part II gives 74 mortality tables based on different subclasses of the population of registration states. Part III gives twelve life tables of foreign countries and ten mortality tables based on experiences of life insurance companies. Part IV presents graphs of life table functions. Part V gives certain life annuities, life insurance premiums and commutation columns. Part VI gives the mathematical theory of the construction of life tables. Part VII gives a detailed account of the process of carrying out the calculations. Part VIII gives tables of the original statistics and an explanation of the different types of data.

The many tables presented in this volume are simply a mine of information for comparative purposes. Part VI on the mathematical theory of the construction of life tables is the part of the work which should prove of much interest to the actuaries interested in the fundamental theory of life table construction. The reviewer believes it correct to say that pages 329–344 give the first clear and detailed statement in the English language of the theory of the continuous flow of population, with applications to the concrete problems of life table construction. The development of these methods centers around the names of Knapp, Zeuner, and Lexis. The author has done an important service in bringing a clear presentation of these methods before American actuaries. In this theory the aggregates of the living and the dead corresponding to assigned birth and age intervals are derived in terms of definite integrals, and the rates of mortality are obtained as the solution of a linear differential equation. The results are made concrete by application to census data.

H. L. Rietz

Espace, Temps, et Gravitation. By A. S. Eddington, translated into French by J. Rossignol, with an introduction by P. Langevin. Paris, J. Hermann, 1921. xii + 262 + 149 pp.

The first part of this volume is a translation of the author's *Space, Time and Gravitation*, reviewed by Professor E. B. Wilson in this BULLETIN (vol. 27 (1921), p. 182). The second part is mathematical and similar to the author's *Report on the Relativity Theory of Gravitation*. The following applies only to the second part.

Eddington has consistently regarded physics as a science where only space-time coincidences or events are observed. These are recorded in terms of coordinates without dimension, then equations are assumed connecting these records and from these equations and records the things of physical significance, such as lengths, times, currents, masses, are *computed*. In this respect he differs from Weyl and most others who regard lengths and times as measured, a procedure which supposes the experimenter endowed with clocks and rods which in some unspecified way adapt themselves to the local geometric requirements.

The exposition is deductive. The postulates include the field equations in empty space and space occupied by matter, and also the differential equations for the motion of a particle. It is possible to derive the field equations by a variation principle where the integrand is the total curvature plus the scalar of the energy tensor. Eddington objects to this because the curvature and energy are physically two aspects of the same thing. On page 76 there is a valuable discussion of the Principle of Equivalence. This is stated in several ways. He uses this principle only a little, "but if we had reasoned by induction, passing from particular laws discovered experimentally to general laws, we would have needed a guiding principle and the Principle of Equivalence would have given us precisely this."

In § 2 there is a much needed treatment of what vectors are in mathematics and of what they are in physics. The last section deals with Weyl's contributions and is characterized by the same vividness and precision as the rest of the book. On page 69, the right-hand member of the second equation of (29, 4) should be negative. This book seems to the reviewer as better adapted than any other to the reader who wishes to spend a limited time on this subject.

K. W. LAMSON

Conférences sur les Transformations en Géométrie Plane. By W. de Tannenberg. Paris, Librairie Vuibert, 1921. 49 pp.

This monograph is concerned with circular transformations, considered primarily as linear transformations of the complex variable. To the beginner it would be of doubtful value, since it assumes the elementary facts concerning the various particular circular transformations and accords scant treatment to the fundamental properties of the general transformation. It deals primarily, both by pure geometry and by analytic methods, with the products of special transformations, e.g., of two rotations or of a rotation and a stretching, and to an advanced student or an instructor interested in this particular subject it should be helpful.

W. C. GRAUSTEIN

Introduction au Calcul Tensoriel et au Calcul Différentiel Absolu. By G. Juvet, with a preface by J. Hadamard. Paris, A. Blanchard, 1922. ii + 101 pp.

This small volume contains a very intelligible account of the so-called "Tensor Theory" which has come to the front with the Einstein relativity. The author starts in with elementary notions of vectors and their linear transformations, then defines tensors algebraically and in connection with bilinear forms, then geometrically. Some consideration of the elementary analysis follows, closing with an exposition of the "parallel displacement" of Levi-Civita, and a general discussion of tensors in a Riemann continuum. There is a good bibliography at the end.

The general point of view is set forth in the interesting preface of Hadamard. His thought reverts back to Poincaré, who said: "The physicist proposes problems to us whose solution he expects. But in making the proposal he pays for the most part in advance for our services. . . . There is an infinite multitude of combinations that may be formed of numbers and symbols. How should one select from this multitude those worthy of attention? Should we be guided by caprice? Such caprice would no doubt alienate our interests far and we would soon cease to understand one another. . . . But there is another side to the matter. . . . Physics not only prevents us from getting lost, but it prevents us from a more serious danger: that of wandering around in a circle."

This situation, Hadamard intimates, had actually arisen in infinitesimal geometry. In fact a crisis had arisen, which fortunately the relativity theory has resolved, through its stimulation of the study of methods already in existence, but mostly ignored, dating back to the absolute geometry of Ricci and Levi-Civita (MATHEMATISCHE ANNALEN, vol. 54).

It is a matter of satisfaction to those of us who have been interested in so-called vector methods, for some time, that they are finally coming into their own. The essential basis of such methods is not the avoidance of coordinates but the development of formulas which are intrinsically given. The present work is open to the criticism to which practically all these investigations are subject, namely that while expressions are produced which are invariant under transformations of the coordinates, there should be no use made of coordinates at all. A proper use of vectors makes the study of absolute geometry not only much simpler and almost obvious, but these same expressions may be translated directly into any desirable system of coordinates with little trouble. The invariancy is a direct consequence of the method. The author remarks in his introduction: "Systems of coordinates are not rejected, and in place of being foreign to the things studied, actually form their structure." Such a point of view is the common one, it is freely admitted, but it is to the detriment of the things studied. It merely shows the path of least resistance taken by the minds of the investigators. When a new generation think as readily in general vectors as many now do in ordinary vectors of three-dimensional space, instead of long demonstrations of invariant and covariant forms, occupying many pages, there will be a few pages of direct statements,

which reach the heart of the matter at once. The real criticism on the use of vectors is that they practically demand that the curved space under consideration be embedded in a flat space of a requisite number of dimensions. This difficulty, however, seems to exist as well for those methods that insist on coordinates. The work of M. Juvet is an example. A perfect intrinsic treatment would never leave the space itself. This can be accomplished by a properly developed system of general vectors. However, when one realizes that if the curved space is embedded in a flat space, then every formula in terms of vectors in the flat space relating to the curved space is *ipso facto* a covariant formula, he will see the whole matter of "theory of tensors" in the proper light.

Hadamard is certainly correct in taking the position that whatever the value of the relativity theory may be, it has done a great thing for geometry in opening up a new life to it, not of the momentary character of a new attack by some geometer, but of the permanent character infused by nature herself. Not only theorems in physics should be stated without systems of axes for reference, but theorems in geometry should also be so stated. The present work will be welcome to those who do think in coordinates, for its clarity and simple presentation.

JAMES BYRNIE SHAW.

Annuaire du Bureau des Longitudes pour l'An 1922. Paris, Gauthier-Villars. 1922. 7 + 800 pp.

There is little to say concerning the current issue of the ANNUAIRE. The review copy arrived somewhat late in the year but this matters less than might be imagined for an almanac, since the data, chiefly astronomical, which change from year to year, are always given in the almanac of the previous year.

There is an attractive little article on Relativity by E. Picard which sets forth the principal points of the theory and the astronomical tests. An article on Money and Exchange by Ch. Lallemand explains the fundamental bases of past and present currencies, and shows the fluctuations of their gold values in some detail during the past eight years.

E. W. BROWN.

Kartenkunde. By M. Groll. Neu bearbeitet von Dr. Otto Graf. I. *Die Projektionen.* Berlin und Leipzig, Vereinigung wissenschaftlicher Verleger, 1922. 116 pp.

This short treatise on map-projections is No. 30 of the well known Sammlung Göschen, and gives a fairly complete account of the numerous systems of mapping of the terrestrial globe. The introduction, which is concerned with general information about drawing, scales, and drawing instruments, and physical geography, is succeeded by four chapters dealing with various projections of the spherical surface upon a plane. In the fifth and last chapter we find a valuable summary and illustrations of the various methods of mapping in use and an historical sketch of their development.

On the whole the little book will be appreciated by students who wish to acquire a general knowledge of map-projections without spending

too much time in studying more ambitious and special treatises on the subject. ARNOLD EMCH.

A Study of Mathematical Education. By Benchara Branford. New edition, enlarged and revised. Oxford, at the Clarendon Press, 1921. xii + 420 pp.

Comparison with the 1908 edition shows the book under review to be a reprint rather than a completely revised edition; according to the preface "the changes in the original, though numerous, are in general of subordinate details," which changes are certainly minor judging from the obvious identity between the two editions even in the indexes. The last 40 pages, or Part III, are entirely new to this edition and under the caption "The Past, the Present, and the Future" discuss the subjects "Adolescent Technique," "Specialists and Cosmology," "Algebra and Mensuration," and "Comparative Algebra, Geometry and Mechanics."

Even though containing little new matter, "the world-wide welcome generously given the work, including a pre-war German translation and a Russian translation in course of preparation," seems to justify this reprint. The work may be characterized as the interesting and thought-provoking meditations on mathematical education of a teacher of twenty or more years' experience in nearly all grades of English schools. The reader feels some lack of unity in the book as a whole which is no doubt due to the fact that it is largely a collection of articles, courses of lectures and addresses written at various times and for various purposes. The following topics and phrases may give some idea of the type and range of the discussions: experimental mathematics, measurement in geometry, types of evidence, educational principles, suggestions from historical developments, nature of geometric knowledge, culture and occupation, presentation determined by ability and background of the learner, good and bad results of the dethronement of Euclid, degree of rigor best fitted to the maturity of the learner, the tongue-tied practical man, danger of discontinuity of logical treatment, need of smaller logically developed systems of propositions, relations of algebra and geometry, suggestions to teachers from non-euclidean geometries and the Einstein theories, the great unities of mathematics, teaching principles as ideals. The progressive teacher sensitive to new ideas and new viewpoints as suggesters in his own thinking and planning will profit by reading this book. ERNEST B. LYTLE

Lezioni di Statistica Metodologica. By Filadelfo Insolera. Turin, Libreria Fratelli Treves, 1921. 191 pp.

This small volume deals with a large variety of topics, including approximate computation, averages, measures of dispersion, permutations, combinations, probability, binomial distribution of frequency, interpolation by the formulas of Newton and Lagrange, graduation of data, least squares, moments, correlation and contingency. The book gives brief elementary expositions of these topics, and will probably serve well its purpose as a means of preparation for certain examinations. On account of the lack of illustrative examples, the reviewer is of the opinion that the book would not be a suitable substitute for certain English and German books for the beginner in the study of statistics. H. L. RIETZ

NOTES

At the meeting of the American Association for the Advancement of Science held at Cambridge, December 26–30, 1922, Professor Harris Hancock of the University of Cincinnati was elected vice-president of Section A, Professor G. A. Miller a member of the Council, and Professor Frank Schlesinger a member of the Committee on Grants for Research in charge of mathematics and astronomy. Professor E. H. Moore delivered his address as retiring president, on the subject, *What is a number system?* Professor Oswald Veblen's address, as retiring vice-president of Section A, was entitled *Geometry and physics*, and was delivered before a joint session of Section A, the American Mathematical Society, and the Mathematical Association of America.

The April, 1922, number (vol. 44, No. 2) of the AMERICAN JOURNAL OF MATHEMATICS contains: *A primary classification of projective transformations in function space*, by L. L. Dines; *A general theory of limits*, by E. H. Moore and H. L. Smith; *Substitution groups whose cycles of the same order contain a given number of letters*, by G. A. Miller; *Boundary value and expansion problems: oscillation, comparison and expansion theorems*, by R. D. Carmichael; *On a theorem in general analysis and the interrelations of eight fundamental properties of classes of functions*, by E. W. Chittenden.

The American Telephone and Telegraph Company has begun publication of a new periodical, THE BELL SYSTEM TECHNICAL JOURNAL, devoted to the scientific and engineering aspects of electrical communication. The two numbers that have appeared contain articles on the mathematical as well as the experimental physics of the telephone and telegraph.

The first number of a new Italian mathematical journal, the BOLLETTINO DELLA UNIONE MATEMATICA ITALIANA, was issued under the date of October, 1922. Professor S. Pincherle, of Bologna, is provisional president of the union.

The Paris Academy of Sciences announces the award of the following prizes for 1922, in addition to those listed in the December number of this BULLETIN (p. 477). Since no memoir was presented on the subject announced for the Grand prize, *The determination of classes of surfaces by given properties of their geodesics* (see this BULLETIN, vol. 26, p. 282), this subject was withdrawn, and the prize awarded to Professor Jean Le Roux, of the University of Rennes, for his mathematical work. The Poncelet prize was awarded to Professor Jules Drach, of the University of Paris, or his mathematical work; the Francœur prize to Dr. Louis Antoine, of the University of Strasbourg, for his work in geometry; the Montyon prize in mechanics to Farid Boulad, of the Egyptian State Railways, for his work in nomography and geometry; the Henri de Parville prize in mechanics to Henri Béghin, of the University of Montpellier, for his

memoir on gyroscopic compasses; the Valz prize to Professor Jean Chazy, of the University of Lille, for his work in celestial mechanics and especially his memoir on the problem of three bodies; the Hughes prize to Camille Raveau, for his work in theoretical physics; the Montyon prize in statistics to Pierre Richard, for his work on the mathematical theory of insurance; the Binoux prize in the history and philosophy of sciences to Professor Gino Loria, for his works on the history of sciences; the Houllevigue prize to Professor Rodolphe Soreau, for his work in aviation and nomography.

The honorary doctorate of the University of Paris has been conferred on Professor A. A. Michelson, of the University of Chicago.

Professor Charles Camichel, of the University of Toulouse, has been elected corresponding member of the Paris Academy of Sciences, in the section of mechanics.

Professor Albert Einstein has been elected a member of the Russian Academy of Sciences.

Professor C. G. Darwin, who is serving as professor of mathematical physics at the California Institute of Technology for the year 1922–23, has been appointed to the newly instituted Tait Chair of Natural Philosophy at the University of Edinburgh.

Associate Professor Emma L. Konantz, of Ohio Wesleyan University, has accepted a permanent appointment at Peking University, effective the second semester of this year. Miss Konantz taught at Peking University while on leave of absence in 1919–21.

Dr. V. E. Pound, of the University of Toronto, has been appointed instructor in mathematics at the University of Buffalo.

Professor G. D. Birkhoff, of Harvard University, will deliver a course of lectures in the graduate school of Yale University during the second half of the present academic year, in the absence of Professor E. W. Brown.

At Columbia University, Professor T. S. Fiske, of the department of mathematics, and Professor C. L. Poor, of the department of celestial mechanics, have been granted leave of absence for the second half year.

Professor Oswald Veblen, of Princeton University, president of the American Mathematical Society, has been granted leave of absence for the second half year.

Professor B. F. Dostal, of the University of Denver, has been appointed professor of mathematics at the Bradley Polytechnic Institute.

Miss Eleanor P. Cushing has been made professor emeritus at Smith College.

NEW PUBLICATIONS

I. PURE MATHEMATICS

Brill (A.). See Salmon (G.).

Fiedler (W.). See Salmon (G.).

Galbrun (H.). Introduction à la théorie de la relativité. Calcul différentiel absolu et géométrie. Paris, Gauthier-Villars, 1923. 8vo 10 + 459 pp.

Grigoras (E. C.). Sur la relativité en algèbre. Bucarest, C. Jonesco, 1922.

Harper (H. D.). See Wentworth (G.).

Heffter (L.). Was ist Mathematik? Unterhaltungen während einer Seereise. Freiburg i. Br., Theodor Fischer, 1922. 160 pp.

Humbert (E.). Traité d'arithmétique. 5e édition. Paris, Vuibert, 1922. 8vo. 508 pp.

Kommerell (K.). See Salmon (G.).

Michaelis (L.). Einführung in die Mathematik für Biologen und Chemiker. 2te, erweiterte und verbesserte Auflage. Berlin, Springer, 1922. 6 + 318 pp.

Mirinny (L.). Isonolyse. (Résolution générale des équations.) Intégration et résolution générale des équations différentielles. Paris, Imprimerie Chaix, 1922. 12 pp.

Salmon (G.) und Fiedler (W.). Analytische Geometrie des Raumes. Unter Mitwirkung von A. Brill neu herausgegeben von K. Kommerell. 1ter Teil: Die Elemente und die Theorie der Flächen zweiter Ordnung. 2te Lieferung. 5te Auflage. Leipzig, Teubner, 1923. 4+369–612 pp.

Scheffers (G.). See Serret (J. A.).

Schuster (A.). Mathematische Unterrichts-Briefe. Band 2: Uebungsbuch. Berlin, Vereinigung Wissenschaftlicher Verleger, 1922. 8vo. 4 + 288 pp.

Serret (J. A.) und Scheffers (G.). Lehrbuch der Differential- und Integralrechnung. 6te und 7te Auflage. Band 2: Integralrechnung. Leipzig, 1921.

Smith (D. E.). See Wentworth (G.).

Struik (D. J.). Grundzüge der mehrdimensionalen Differentialgeometrie in direkter Darstellung. Berlin, Springer, 1922. 8 + 198 pp.

Wentworth (G.), Smith (D. E.), and Harper (H. D.). Fundamentals of practical mathematics. Boston, Ginn, 1922. 5 + 202 pp. $1.50.

Yule (G. U.). An introduction to the theory of statistics. 6th edition, enlarged. London, C. Griffin, 1922. 8vo. 430 pp.

II. APPLIED MATHEMATICS

BROSE (H. L.). See MOSKOWSKI (A.).

DUFRENOIS (—.), RISSER (—.), et ROUSIER (—.). Les méthodes actuelles de la balistique extérieure. Paris, Gauthier-Villars, 1921. 10 + 244 pp.

DUVAL (A. B.) et HÉBRARD (L.). Traité pratique de navigation aérienne. Paris, Gauthier-Villars, 1922. 4to. 60 pp.

EDDINGTON (A. S.). Exposé théorique de la relativité généralisée. Complément mathématique inédit de l'édition française d'Espace, temps et gravitation. Paris, Hermann, 1922. 4to. 4 + 149 pp.

EINSTEIN (A.). Untersuchungen über die Theorie der "Brownschen Bewegung." Mit Anwerkungen herausgegeben von R. Fürth. (Ostwald's Klassiker der exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1922. 8vo. 72 pp.

———. See LORENTZ (H. A.).

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. Band V 1, Heft 6: K. F. Herzfeld, Physikalische und Elektrochemie. Leipzig, Teubner, 1921.

EXNER (F.). Vorlesungen über die physikalischen Grundlagen der Naturwissenschaften. 2te Auflage. Leipzig und Wien, Deuticke, 1922. 20 + 734 pp.

FABRY (C.). Eléments d'électricité. Paris, Colin, 1921. 208 pp.

FISHER (I.). The making of index numbers. A study of their varieties, tests, and reliability. Boston, Houghton Mifflin, 1922. 32 + 526 pp.

FÖPPL (A.). Vorlesungen über technische Mechanik. Band 5 und 6. 4te Auflage. Leipzig, 1921–1922.

FÜRTH (R.). See EINSTEIN (A.).

GANDILLOT (M.). Véritable interprétation des théories relativistes. Paris, Gauthier-Villars, 1922. 16mo. 17 pp.

GIBSON (A. H.). Hydro-electric engineering. Volume 1: Civil and mechanical engineering. London, Blackie, 1921. 10 + 232 pp.

GRAETZ (L.). Handbuch der Elektrizität und des Magnetismus. In fünf Bänden. Herausgegeben von L. Graetz. Leipzig, Barth. Band 1: Elektrizitätserregung und Elektrostatik. 1918. 8 + 760 pp. Band 2: Stationäre Ströme. 1921. 10 + 772 pp. Band 4: Magnetismus und Elektromagnetismus. 1920. 10 + 1360 pp.

GRIMSEHL (E.). Lehrbuch der Physik. Band 1: Mechanik, Wärmelehre, Akustik und Optik. 5te Auflage. Leipzig, Teubner, 1921. 16 + 1029 pp.

GRUNER (P.). Leitfaden der geometrischen Optik und ihrer Anwendungen auf die optischen Instrumente. Bern, 1921.

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HAUBER (W.). Statik. Teil 1: Die Grundlehren der Statik starrer Körper. 6ter Neudruck. Berlin, 1921.

HÉBRARD (L.). See DUVAL (A. B.).

HELMHOLTZ (H. VON). Vorlesungen über theoretische Physik. Band 6:

Vorlesungen über die Theorie der Wärme. 2ter, durchgesehener Abdruck. Leipzig, Barth, 1922. 12 + 418 pp.

HENNING (F.). See SEELIGER (R.).

HENTSCHEL (W.). Das Relativitätsprinzip in Rahmen einer Gesamtansicht von Welt und Mensch. Leipzig, E. Matthes, 1921. 8vo. 77 pp.

HERZFELD (K. F.). See ENCYKLOPÄDIE.

HUGUET (L.) et SUFFRIN-HÉBERT (M.). Calculs aéro-dynamique des avions. Lois de résistance de l'air. Paris, Béranger, 1922. 16mo. 120 pp.

KIRCHBERGER (P.). Die Entwicklung der Atomtheorie. Karlsruhe, Müller, 1922. 8vo. 10 + 260 pp.

KOSSEL (W.). Valenzkräfte und Röntgenspektren. Zwei Aufsätze über das Elektronengebäude des Atoms. Berlin, Springer, 1921. 70 pp.

LANGEVIN (P.). See THOMPSON (J. J.).

LEBEL (J.). Résumé des principes de la perspective linéaire. Paris, Vuibert, 1921. 48 pp.

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PETZOLDT (J.). Das Weltproblem vom Standpunkte des relativistischen Positivismus aus, historisch-kritisch dargestellt. 3te, neubearbeitete Auflage, unter besonderer Berucksichtung der Relativitätstheorie. (Wissenschaft und Hypothese, Band 14.) Leipzig, Teubner, 1921.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

THE SAN FRANCISCO SECTION, at STANFORD UNIVERSITY, April 7, 1923.

Abstracts must be in the hands of the Secretary of the Section, B. A. Bernstein, not later than March 24.

CHICAGO, April 13 and 14, 1923. See note, p. 476, vol. 28.

Abstracts must be in the hands of the Secretary of the Chicago Section, Arnold Dresden, not later than March 23.

NEW YORK CITY, April 28, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than April 14.

R. G. D. RICHARDSON,
Secretary of the Society.

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